Teaching square roots: Conceptual complexity in mathematics language

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Mathematics is an "artificial" deliberately constructed language, supported crucially by:

- special alpha-numeric characters and usages, including:
  - letters of the standard English alphabet, A, a, B, b, … Z, z, printed and cursive;
  - occasionally the Greek alphabeta — such as α, β, Δ, ϕ, Σ and μ;
  - more rarely the Hebrew alephbeth — such as aleph, $\aleph$;
  - and occasionally German gothic capital letters — such as $\Im$, $\Re$; with
  - Hindu-Arabic digits or numerals, 0, 1, 2, 3, … 9; and
  - occasional use of Roman numerals — I, II, III, and i, ii, iii, …;
- extra-special non-alphanumeric symbols — such as $\surd$, $\times$, $\sim$, $\div$, Σ, ∞ and even $\forall$ — and degree, minute (foot) and second (inch) symbols;
- special written formats within a single line, such as superscripts and subscripts;
- grouping along a line, including bracketing using round brackets, (…), parentheses, […], and braces, {…}; and
- the clever use of two or more lines at a time (as in fraction notation), and the set-theoretic and logical connectives such as &, $\Rightarrow$, $\in$, $\leftrightarrow$, $\cup$, $\neg$.

Also in important *non-verbal* ways this "language" depends crucially on spatial-textual formatting devices, and non-verbal images, including tables with rows and columns, pictures, diagrams, and graphs, and spatial-visual conventions. These include isometric diagrams, angle-notch markings to show equal angles at the base of an isosceles triangle, chevron or arrowhead markings to show pairs of parallel lines, line-notch markings to show pairs of equal-length sides, arrowheads showing directions on axes of graphs, or on vectors and compass bearings (see Barling's (2005) discussion of specialised mathematical text and symbolic formatting in our computer keyboard and CAS era).

In teaching mathematics we surround the focal formal content language which is mathematics, with other informal instructional language. Obviously, in English-speaking schools, English is this language of instruction. Inevitably there is an unavoidable overlap between the language of instruction and the

language being taught. For English speakers, many mathematical words are taken directly from everyday English, fitted with specialised meanings, and used, from then on, with special mathematical significance (because English and English-versions of mathematics also borrow heavily from non-English languages, we can have multilingual complexities).

For example, "volume" in mathematics is not the turning-knob or slider-control on the stereo-system or television which we adjust to make the sound louder, nor is it a separately bound part of a multi-book publication. Of course, there are semantic (meaningful) connections between the amount of sound (or loudness), and the amount or quantity of three-dimensional space which some object occupies. In the formal context of mathematics instruction, the technical distinctions between "volume" and "capacity" may not always be clear (McDonough, 2004). The total space occupied by a solid, filled and/or hole-less, three-dimensional object may be distinct from the possible "fillability" of a "hollow" but "solid-walled" three-dimensional object, and contrasts with the physical space of the "walls" of the potentially fillable object. These ideas are confusing for primary children working physically with jugs, cans, boxes, centicubes and MAB materials, and for calculus students integrating "solids" of revolution. Consider the example of McDonough's "box with thick walls" (McDonough, 2004, p. 328) that can be made from MAB base-6 and base-5 flats: use a base or floor with dimensions 6 cm × 6 cm × 1 cm, and four walls each of 5 cm × 5 cm × 1 cm. The external size of the hollow-cube is $6 \times 6 \times 6$ cm^3, the wall-and-floor "volume" is 136 cm^3, and the inner "capacity" is 80 cm^3.

When learning new words, symbols, visual conventions, operations and procedures in mathematics, students must learn to distinguish the new and specific mathematical meanings from any familiar, looser, everyday meanings that might be confused with its mathematical use. Students should also find ways of expressing these new highly specialised meanings in confident personally fluent ways, possibly re-expressing these new words, symbols, diagrams and processes in their own careful choices of non-mathematical everyday words. We not only need to learn how to speak and do mathematics, within mathematics-as-the-language, we also need to learn how to talk in non-mathematical language about what we are saying and doing in mathematics.

Douglas Barnes, in the 1986 edition of *Language, the Learner and the School*, comments that:

> A pupil's understanding of a new topic depends upon bringing what he or she knows already to bear upon it, since our ability to understand a message depends on the resources we bring to it. Meaning does not lie in words but in the cultural practices of those who use them… The pedagogical problem in a room full of pupils is how to enable all of them to bring to mind relevant knowledge and understanding, and to "recode" it in terms of the new framework offered by the teacher. That is why pupils' participation in the formulating of ideas in speech or writing is of crucial importance (Barnes, Britton and Torbe, 1986, p. 26).

Without Barnes actually using the word, this is an early account of what is now referred to as "constructivism" or "social constructivism" — the metaphor that describes meaningful learning. Here, "recoding" is partly the process of constructing personal, and social, meaning, building on, adjusting, and adding to existing knowledge.

Consider the example of "square root" (or "quadratwurzel" in German!). When first heard, it leaves the non-mathematically trained English-speaker none the wiser. If we can "square" or "cube" a number, why cannot we "corner," "diagonal," "triangle," or "pyramid" it? Then if we succeed in grasping that "getting the square root of a number" is some kind of "reverse-square", why cannot we "reverse-triangle," or "reverse sphere," for example? (Maybe we can!)

Incidentally the odd German word for "square root"(and equivalents in other European languages), like the English term, literally translates the earlier and equally odd Latin: "quadtratix". For a German student, what does a "quadrilateral" have to do with a "wurzel" or mangold-wurzel (a turnip-like plant in the general "beet" family, such as beetroot)? The word "beetroot" is itself a linguistic oddity, as the "root" part of the word actually indicates "red" or "rot" in German, rather than the apparent plant-term "root"; that is, a "beetroot" is a "red beet", not a "beet with a root". The misleading plant-root meaning does not function as a distinguishing feature of the plant because all "beets" have roots, including sugar beets, silver beet and others, distinguished by being sweet or by their alleged silver colour.

The original Latin "radix", which appears in "quadratix" and transformed into "root", indicates a process of extraction, metaphorically comparable to digging a plant up by its roots. Hence comes the idea of taking the root of a square: extracting or getting the side (an odd sort of "root"), if we know the square, or quadrangle. Unfortunately in ordinary English (and German) "root" (or "wurzel") does not carry much meaning of "extracting" or "side". The everyday plant, or gardening, connotations (to say nothing of the cruder vernacular connotations) fail to indicate anything sensible at all about the intended mathematical meaning.

What we actually intend by the term "square root" is something like this:

> Start with a number, possibly any (positive) number, at all. Imagine that it is, in some way, expressible as a "square". Then find the length of the side of that square.

Instead of a simple definition, or descriptive fact-statement, we have a not immediately obvious backwards process, which, importantly, reverses a process of "squaring" which is assumed to have been taught and learned beforehand. When students first encounter "square root," they are expected to pull together geometric ideas (recognising a square shape) and numerical ideas (a number multiplied together with itself), and then mentally re-work the process of squaring so it becomes a new, but related, reverse process of "treat it like a square and find the side". As such, students need to be able to "tell the story" of what "square root" means, or to know the "gossip" about the

term and the associated process (Barnes, 1976; Callingham, 2004, p. 11, citing Devlin, 2000).

Students have to "encapsulate" the component ideas of this new concept (Gray & Tall, 1994), or "unitise" (Steffe & Cobb, 1988). Here "unitising" is the conceptual act of forming a single new whole "thing" from what had previously been a collection of discrete unitary elements. A related learning-construction tool is "disembedding": the conceptual act of dissociating or deconstructing sub-units from a composite unit while still recognising the original unity of the concept-whole.

Both conceptually, and in terms of *making* a square root, there is a major problem: if we start with a number (any one), how can it meaningfully be "turned into a square"?

By the time most students meet square roots, they have already begun thinking of numbers as points or distances along a number "line". They also know or believe that a "line", geometrically, has no width. To overcome the mental barriers inherent in these more advanced concepts, students might be helped by reverting to a conceptually earlier and more primitive model of numbers, where the numbers are linear collections of discrete wholes; e.g., 9 can be represented as a strip of stars or dots:

• • • • • • • • •

It is then easy to re-organise a line of nine dots (•) into a different geometric shape, at which stage the "square-ness" and "side-ness", and their numerical equivalents, are obvious; e.g.,

• • •
• • •
• • •

Playing with figurate numbers can help making triangle and other geometric number-shapes. However, this does not so easily work with, for example, 8, 10 or 9.5. Students encountering the idea of a square root for the first time have had some experience of turning a length into an area by a visually constructive process of multiplying, including the spatial equivalent of making a rectangle or square using the two factors as the sides of the figure; but they have had little experience with, or practical need (if ever) to taking a *length* and turning that directly into a square *area.* Now they have to.

Suppose, for example, we take this as a model of 1

$\boxed{1}$

and then this is a model of, say, 10

1	1	1	1	1	1	1	1	1	1

We might move towards a clearer idea of what is meant by finding a square root if we started slicing up such a paper-strip model of a given number to make a square. This would, incidentally, prepare for later developing the alge-

braic and spatial-visual process of completing a square, in working with quadratic expressions, as well as preparing for the idea of algorithmic processes used to make step-by-step approximations to a square root (such as the Newton method: see Meserve, Sobel & Dossey, 1987, p. 275).

For example, a first approximation to finding the square root of 10, using a paper-strip model, would be to construct the next smaller square, namely 9:

1	1	1
1	1	1
1	1	1

Then, seeing the remaining portion (1), consider how this might be sliced up, into (six?) thinner strips, and placed alongside the 3 × 3 square… and so on.

For all students, discussing everyday examples is essential, wherever possible, to demonstrate how the technical term is used in practice, if it is used at all in everyday mother-tongue ways. Algebra, for example, is not famous for its everyday uses. Such discussion of examples is often essential to motivate students, and to make opaque, difficult, unfamiliar, unnatural or counter-intuitive ideas sensible. Naturally it is best handled, as much as possible, in the local language, rather than in the language of instruction. How much do you encourage your students to talk with each other, and you, about what they are learning?

References and further reading

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